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A note concerning free-convective boundary-layer flows

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Summary

We show that the free-convective boundary-layer flow in a porous medium in the vicinity of a parameter value at which similarity solutions cease to exist is of semi-jet form.

1. The equation and analysis

The equation

$$f''' + ff'' - \beta f'^2 = 0 \tag{1}$$

with boundary conditions

$$f(0) = 0, \quad f''(0) = -1, \quad f'(\eta) \to 0 \text{ as } \eta \to \infty \tag{2}$$

arises in a problem involving free convection in a porous medium, and numerical solutions have been given for values of β in the range $-1 < \beta \leqslant 1$ by Merkin [1]. One can infer from the numerical results that as $\beta \downarrow -1$ both $f(\infty)$ and $f'(0)$ are becoming large, and indeed Merkin has shown that when $\beta = \beta^* = -1$ there is no solution of (1) subject to (2). This behaviour has some similarities to that found by Banks [2] who investigated solutions of (1) subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\eta) \to 0 \text{ as } \eta \to \infty,$$

and which governs the flow near a stretching impermeable wall. The critical value for β in this case is -2 and the structure of $f(\eta)$ in the neighbourhood of $\beta = -2$ was presented in [2]. The purpose of this brief note is to present the analogous structure of $f(\eta)$ as defined by (1) and (2) for the case when $0 < 1 + \beta \ll 1$.

With $\epsilon = 1 + \beta > 0$ we look for a solution of (1) by writing

$$f(\eta) = \epsilon^p F(z), \quad z = \epsilon^q \eta. \tag{3}$$

A solution in which there is a balance between all three terms in equation (1) is possible provided that $p = q$ and that $F(z)$ satisfies

$$F''' + FF'' + (1 - \epsilon) F'^2 = 0, \tag{4}$$

where dashes now signify differentiation with respect to z. The second relationship between p and q is obtained by integrating (1) across the boundary layer to get

$$(1+\beta)\int_0^\infty f'^2 \mathrm{d}\eta = 1, \tag{5}$$

if we assume a suitable decay rate for f' as $\eta \to \infty$. Substitution of (3) into (5) yields the relationship $2p+q=-1$ and so $p=q=-1/3$. The boundary conditions become

$$F(0)=0, \quad F''(0)=-\epsilon, \quad F'(z)\to 0 \text{ as } z\to\infty, \tag{6}$$

while the integral (5) becomes

$$\int_0^\infty F'^2 \mathrm{d}z = 1. \tag{7}$$

By first writing equation (1) in terms of von Mises variables, where $f'(\eta)$ is regarded as a function of f, Merkin gave an approximate method of solution which gives very good agreement with the numerical results. It is also possible to deduce from the approximate analysis that, as in the exact treatment given here, $f(\eta) \sim \epsilon^{-1/3}$ and $f'(0) \sim \epsilon^{-2/3}$.

We now look for a solution of (4) subject to (6) and (7) by writing

$$F(z) = F_0(z) + \epsilon F_1(z) + O(\epsilon^2) \tag{8}$$

and find

$$F_0''' + F_0F_0'' + F_0'^2 = 0, \tag{9}$$

$$F_1''' + F_0F_1'' + 2F_0'F_1' + F_0''F_1 = F_0'^2, \tag{10}$$

with boundary conditions

$$F_0(0) = F_0''(0) = 0, \quad F_0'(z)\to 0 \text{ as } z\to\infty, \tag{11}$$

$$F_1(0)=0, \quad F_1''(0) = -1, \quad F_1'(z)\to 0 \text{ as } z\to\infty. \tag{12}$$

The solution of (9) which satisfies the boundary conditions (11) and the constraint $\int_0^\infty F_0'^2 \mathrm{d}z = 1$ from (7) is

$$F_0 = a\tanh(az/2), \tag{13}$$

where $a = 3^{1/3}$. It is of interest to note that the leading term found here coincides with a half of the well-known two-dimensional free-jet solution, whereas in the analogous problem considered by Banks, referred to above, the limiting flow was of wall-jet type.

It is also possible to find the next term in (8) without too much labour. We find that the solution of (10) which satisfies the boundary conditions in (12) is

$$F_1 = a\,\mathrm{sech}^2\zeta \int_0^\zeta \{\tfrac{4}{3}\cosh^2\theta(\log\cosh\theta - \theta) - \tfrac{1}{3} + B\cosh^2\theta\}\mathrm{d}\theta,$$

where $\zeta = az/2$ and B is a constant. The latter is determined by satisfying the constraint in (7) which becomes, to this order, $\int_0^\infty F_0' F_1' \mathrm{d}z = 0$, and on carrying out the details we obtain

$$B = (24 \log 2 - 10)/9.$$

The salient properties given by Merkin are $f'(0)$ and $f(\infty)$ which, for $0 < 1 + \beta \ll 1$, we have shown can be written

$$f'(0) = \epsilon^{-2/3}\{a^2/2 + [a^2(24 \log 2 - 7)/18]\epsilon + O(\epsilon^2)\}$$

and

$$f(\infty) = \epsilon^{-1/3}\{a + [a(6 \log 2 - 5)/9]\epsilon + O(\epsilon^2)\}.$$

For $\beta = -0.9$ Merkin gives $f'(0) = 5.016$ and $f(\infty) = 3.079$ whereas the two terms of the asymptotic theory given here lead to 5.022 and 3.078 respectively.

It is now possible to proceed, as did Banks, to situations in which $\beta \leqslant -1$, and argue that the asymptotic flow in this régime is related to the semi-jet form given in (13) albeit with a different argument. Full details are given in [2] and indeed a full numerical investigation of the problem of a stretching wall is in progress in which the similarity solutions are linked to flows of non-similarity type.

References

[1] J.H. Merkin, A note on the solution of a differential equation arising in boundary-layer theory, *J. Eng. Math.* 18 (1984) 31–36.

[2] W.H.H. Banks, Similarity solutions of the boundary-layer equations for a stretching wall, *J. de Mécan. théor. et appl.* 2 (1983) 375–392.